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# THERMODYNAMICS OF FINITE STRAIN ELASTIC–INELASTIC DEFORMATION

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*The kinematic relations describing elastic–inelastic deformation that coincide in shape with the well-known Lie representation but are free from the drawback of the latter are extended to the case of thermo-elastic–inelastic deformation with finite strains. The limitations imposed on the kinematics by the principle of objectivity are considered. Relations for the stresses and entropy are derived from the laws of thermodynamics, and a heat-conduction equation is constructed.*

**Key words:** *thermo-elastic–inelastic finite strains, kinematics, thermodynamics, principle of objectivity.*

**1. Kinematic Relations.** An approach to constructing the constitutive equations of complex media for finite elastic–inelastic deformation was proposed in [1–3]. The kinematics of the process is described by a relation which takes into account the real history of elastic–inelastic deformation, i.e., any sequence and duration of purely elastic and purely inelastic deformations:

$$F = f \cdot F_*. \tag{1.1}$$

Here $F$, $f$, and $F_*$ are the elastic–inelastic site gradients which transform the initial configuration to the current one, an intermediate configuration close to the current one to the actual configuration, and the initial configuration to the intermediate one. Because of the similarity between the intermediate and current configurations, $f = f_E \cdot f_{IN} = f_{IN} \cdot f_E$ ($f_E$ and $f_{IN}$ are the elastic and inelastic site gradients, respectively). In [3], the inelastic ($F_{IN}$) and purely elastic ($F_E$) kinematics were extracted from the kinematics (1.1) using the concepts of a matricant and a multiplicative integral. As a result, relation (1.1) is represented as

$$F = F_E \cdot F_{IN}, \tag{1.2}$$

where all site gradients are determined at the current time $t$. Representation (1.2) coincides in form with the well-known Lie representation but it is free from the drawbacks of the latter. In particular, from this representation, it follows that the total displacement rate deformation $D$ is the sum of the elastic rate deformation $D_E$ and the inelastic rate deformation $D_{IN}$, the elastic site gradient $F_E$ remains unchanged under purely inelastic changes in the configuration, and the inelastic site gradient remains unchanged under its purely elastic changes.

The expressions for $F_E$ and $F_{IN}$ obtained in [3] have the form

$$F_E = (g + \varepsilon h_E) \cdot F_{E*}; \tag{1.3}$$

$$F_{IN} = (g + \varepsilon F_{E*}^{-1} \cdot h_{IN} \cdot F_{E*}) \cdot F_{IN*}. \tag{1.4}$$

Here $g$ is the unit tensor; the site gradients with the subscript asterisk correspond to the time $t_*$, and the site gradients without the subscript corresponding to the current time $t$ ($t - t_* = \varepsilon\tau$, where $\tau > 0$ and $\varepsilon$ is a small positive parameter); $h_E$ and $h_{IN}$ are the elastic and inelastic displacement gradients with respect to the configuration defined by $F_*$, which are represented in terms of the symmetric part $e_E$ and $e_{IN}$ (small strains) and the skew-symmetric

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part $d_E$ and $d_{IN}$ (small rotations), and $e = e_E + e_{IN}$ and $d = d_E + d_{IN}$ are the total small strains and rotations (these and only these strains and rotations are compatible).

According to relation (1.2), the Cauchy–Green strain measure $C = F^{\rm t} \cdot F$ is written as $C = F_{IN}^{\rm t} \cdot C_E \cdot F_{IN}$, where $C_E = F_E^{\rm t} \cdot F_E$. In view of (1.3) and (1.4), this measure can be represented as

$$C = C_* + 2\varepsilon F_*^{\rm t} \cdot (e_E + e_{IN}) \cdot F_*, \qquad F_* = F_{E*} \cdot F_{IN*}, \qquad C_* = F_*^{\rm t} \cdot F_* \tag{1.5}$$

or

$$C = C_\star + 2\varepsilon F_\star^{\rm t} \cdot e_E \cdot F_\star, \qquad F_\star = F_{E\star} \cdot F_{IN}, \qquad C_\star = F_\star^{\rm t} \cdot F_\star. \tag{1.6}$$

Here the quantities with the subscript $*$ correspond to the intermediate configuration $\varkappa_1$, and the quantities with the subscript $\star$ to the intermediate elastic configuration $\varkappa_2$ (see the figure in [3]) at the same time $t_*$. According to these relations, as the intermediate configuration $\varkappa_1$ tends to the current configuration ($F_* \to F$ and $C_* \to C$) and as the intermediate elastic configuration $\varkappa_2$ tends to the current configuration ($F_\star \to F$ and $C_\star \to C$), the passage to the limit gives two increments and two rates of change in the strain measure $C$:

$$(dC)_{\varkappa_1} = 2F^{\rm t} \cdot (de_E + de_{IN}) \cdot F,$$

$$(\dot{C})_{\varkappa_1} = 2F^{\rm t} \cdot (\dot{e}_E + \dot{e}_{IN}) \cdot F = 2F^{\rm t} \cdot (D_E + D_{IN}) \cdot F$$

with respect to the configuration $\varkappa_1$ (the total increment and the total rate of change in the tensor $C$) and

$$\begin{gathered} (dC)_{\varkappa_2} = 2F^{\rm t} \cdot de_E \cdot F, \\ (\dot{C})_{\varkappa_2} = 2F^{\rm t} \cdot \dot{e}_E \cdot F = 2F^{\rm t} \cdot D_E \cdot F \end{gathered} \tag{1.7}$$

with respect to the configuration $\varkappa_2$ (the increment and rate of change in the tensor $C$ due to only elastic deformation). Therefore, the tensor given by relation (1.5) will be denoted by $C_{\varkappa_1}$, and the tensor given by relation (1.6) by $C_{\varkappa_2}$. In the relations for the increments and rates, $D_E = \dot{e}_E$ and $D_{IN} = \dot{e}_{IN}$ are the rate deformation tensors for the corresponding displacements, which in this case coincide with the strain rate tensors.

As in [3], the temperature effect is taken into account by representing the kinematics of the thermo-elastic–inelastic process as $F = f_E \cdot f_{IN} \cdot f_\Theta \cdot F_*$, where $f_\Theta$ is the site gradient that correspond to small temperature strains; $F_*$ is the thermo-elastic–inelastic site gradient that transform the initial configuration to the intermediate one. In this case, all site gradients given by small strains commute with each other. As in [3], we obtain

$$\begin{gathered} F = F_E \cdot F_{IN} \cdot F_\Theta = [g + \varepsilon(h_E + h_{IN} + h_\Theta)] \cdot F_*, \\ F_* = F_{E*} \cdot F_{IN*} \cdot F_{\Theta*}. \end{gathered} \tag{1.8}$$

Here $F_E$ and $F_{IN}$ are given by relations (1.3) and (1.4) and $h_\Theta$ is the temperature rate gradient with respect to the configuration $F_*$;

$$F_\Theta = (g + \varepsilon F_{IN*}^{-1} \cdot F_{E*}^{-1} \cdot h_\Theta \cdot F_{E*} \cdot F_{IN*}) \cdot F_{\Theta*}. \tag{1.9}$$

(We note that in relations (1.8) and (1.9), the subscripts $IN$ and $\Theta$ can be interchanged.) As a result, the total small strains and rotations are given by the expressions $e = e_E + e_{IN} + e_\Theta$, $d = d_E + d_{IN} + d_\Theta$, where $e_\Theta$ and $d_\Theta$ are the symmetric and skew-symmetric parts of $h_\Theta$.

Similarly to [3], it is easy to show that the total displacement rate deformation $D$ is the sum of the elastic rate deformation $D_E$, the inelastic rate deformation $D_{IN}$, and the temperature rate deformation $D_\Theta$; the elastic site gradient remains unchanged under purely inelastic and temperature changes in the configuration, the inelastic site gradient under purely elastic and temperature in the configuration, and the temperature site gradient under purely elastic and inelastic changes. The increment and rate of change in the Cauchy–Green strain measure $C$ with respect to the intermediate elastic configuration $\varkappa_2$ is given by relation (1.7), in which the total site gradient is defined by expression (1.8).

**2. Relations Implied by the Laws of Thermodynamics.** We write the thermodynamic Clausius–Duhem inequality as

$$T \cdot\cdot D - \rho(\dot{\Psi} + \dot{\Theta} s) - \boldsymbol{q} \cdot \tilde{\nabla} \ln \Theta \geqslant 0,$$

where $\rho$, $\Psi$, and $s$ are the mass density in the current configuration and the specific (referred to unit mass) free energy and entropy, $\boldsymbol{q}$ is the heat flux vector, $\tilde{\nabla}$ is the Hamilton operator in the current configuration, and $D = \dot{e}_E + \dot{e}_{IN} + \dot{e}_\Theta$ is the tensor of total displacement rate deformation. According to the principle of objectivity, the arguments of the function $\Psi$ can be only invariant quantities, i.e., only any kinematic quantity invariant under rigid rotation of the current configuration, the temperature $\Theta$, and a finite number of internal parameters $\chi_i$ ($i = 1, \ldots, m$) — objective scalar functions that characterize the change in the internal structure of the material during elastic–inelastic deformation. As the kinematic quantity, we use the tensor $C$ and represent the specific free energy $\Psi = \Psi(C, \chi_i, \Theta)$ as $\Psi(C, \chi_i, \Theta) = \Psi_1(C, \chi_i, \Theta) + \Psi_2(\Theta)$, assuming that: 1) $\dot{\Psi}_1 = 0$ if $(\dot{C})_{\varkappa_2} = 0$ ($(dC)_{\varkappa_2} = 0$; 2) $\Psi_2 = 0$ if $\Theta = \Theta_0$. Here $\Theta_0$ is the reduction temperature in Kelvin (usually, room temperature). According to the first condition [see (1.7)], if there is no change in the elastic strain ($\dot{e}_E = D_E = 0$), then $\Psi_1$ also remains unchanged. Therefore, any elastic–inelastic process is treated as an elastic process with a stressed reference configuration and is modeled by a series connection of elastic, inelastic, and temperature elements. As is assumed in many papers (see, for example, [4–8]), the quantity $\Psi_1$ is the energy accumulated in the elastic element. The first condition is satisfied by the functional $W_1$ introduced in [3]:

$$W_1 = \int\limits_0^t \Big( \int\limits_0^\tau \frac{\partial^2 W}{\partial C_E^2} \cdot\cdot\, \dot{C}_{\varkappa_2}\, d\tau_1 \Big) \cdot\cdot\, \dot{C}_{\varkappa_2}\, d\tau. \tag{2.1}$$

Here $W$ is the elastic potential which depends only on the elastic kinematics, which, in turn, depends on $C_{\varkappa_2}$ [see relation (1.6)]. If we assume that the constants $a_k$ ($k = 1, \ldots, n$) of this elastic potential are functions of the inelastic kinematics and temperature [$a_k = a_k(\chi_i, \Theta)$], the independent variables in the functional $W_1$ will be $C_{\varkappa_2}$, $\chi_i$, and $\Theta$. Then,

$$\dot{W}_1 = \frac{\partial W_1}{\partial C_{\varkappa_2}} \cdot\cdot\, \dot{C}_{\varkappa_2} + \frac{\partial W_1}{\partial \chi_i}\, \dot{\chi}_i + \frac{\partial W_1}{\partial \Theta}\, \dot{\Theta} = \Big( \int\limits_0^t \frac{\partial^2 W}{\partial C_E^2} \cdot\cdot\, \dot{C}_{\varkappa_2}\, d\tau \Big) \cdot\cdot\, \dot{C}_{\varkappa_2}$$
$$+ \int\limits_0^t \Big[ \int\limits_0^\tau \Big( \frac{\partial^3 W}{\partial \chi_i\, \partial C_E^2}\, \dot{\chi}_i + \frac{\partial^3 W}{\partial \Theta\, \partial C_E^2}\, \dot{\Theta} \Big) \cdot\cdot\, \dot{C}_{\varkappa_2}\, d\tau_1 \Big] \cdot\cdot\, \dot{C}_{\varkappa_2}\, d\tau. \tag{2.2}$$

If we set $W = W(C_E(\tau), \chi_i(t), \Theta(t))$, i.e., if the elastic potential $W$ contains the functions $\chi_i$ and temperature $\Theta$ as parameters dependent on the current time $t$, this result can be obtained by direct differentiation of the functional (2.1) with respect to $t$. With the use of relation (1.7), this functional can be written as

$$W_1 = 4 \int\limits_0^t \Big\{ F \cdot \Big[ \int\limits_0^{\tau_1} \Big( F \overset{3}{\circ} \frac{\partial^2 W}{\partial C_E^2} \cdot F^{\mathrm{t}} \Big) \cdot\cdot\, D_E\, d\tau_2 \Big] \cdot F^{\mathrm{t}} \Big\} \cdot\cdot\, D_E\, d\tau_1. \tag{2.3}$$

Here the sign "$\overset{3}{\circ}$" denotes scalar premultiplication of the second rank tensor $F$ into the third basis vector of the fourth-rank tensor $\partial^2 W / \partial C_E^2$.

The functional $W_1$ is referred not to unit mass but to unit undeformed volume; therefore, $\rho_0 \Psi_1 = W_1$ and $\Psi = W_1/\rho_0 + \Psi_2(\Theta)$, where $\rho_0$ is the mass density in the initial configuration. Since $\rho = J^{-1}\rho_0$ ($J$ is the Jacobian which defines the relative volume change), then, substituting the expressions for $D$, $\dot{\Psi}$, $\rho$ into the Clausius–Duhem inequality, we have

$$\Big( T - 2J^{-1} F \cdot \frac{\partial W_1}{\partial C_{\varkappa_2}} \cdot F^{\mathrm{t}} \Big) \cdot\cdot\, \dot{e}_E + T \cdot\cdot\, \dot{e}_{IN} + T \cdot\cdot\, \dot{e}_\Theta$$
$$- J^{-1} \frac{\partial W_1}{\partial \chi_i}\, \dot{\chi}_i - J^{-1} \rho_0 \Big( \frac{1}{\rho_0} \frac{\partial W_1}{\partial \Theta} + \frac{\partial \Psi_2}{\partial \Theta} + s \Big) \dot{\Theta} - \boldsymbol{q} \cdot \tilde{\nabla} \ln \Theta \geqslant 0. \tag{2.4}$$

We construct a local continuation of the process [9] and relate $\dot{e}_\Theta$ with the variation in the temperature $\dot{\Theta}$ by the simple law of linear temperature expansion $\dot{e}_\Theta = \beta \dot{\Theta} g$, where $\beta$ is the linear temperature expansion coefficient, which is assumed to be a function of only the temperature. As a result, for the derivative $\partial W_1 / \partial C_{\varkappa_2}$, in view (2.2), we obtain

$$T = J^{-1} F \cdot P_{\mathrm{II}} \cdot F^{\mathrm{t}},$$

$$P_{\mathrm{II}} = 2\int_0^t \frac{\partial^2 W}{\partial C_E^2} \cdot\cdot\, \dot{C}_{\varkappa_2}\, d\tau = 4\int_0^t \Big(F \overset{3}{\circ} \frac{\partial^2 W}{\partial C_E^2} \cdot F^{\mathrm{t}}\Big) \cdot\cdot\, D_E\, d\tau, \tag{2.5}$$

where $P_{\mathrm{II}}$ is the symmetric Piola–Kirchhoff stress tensor,

$$s = J\,\frac{\beta}{\rho_0}\, I_1(T) - \frac{1}{\rho_0}\,\frac{\partial W_1}{\partial \Theta} - \frac{\partial \Psi_2}{\partial \Theta}, \qquad T \cdot\cdot\, \dot{e}_{IN} - J^{-1}\frac{\partial W_1}{\partial \chi_i}\,\dot{\chi}_i - \boldsymbol{q} \cdot \tilde{\nabla} \ln \Theta \geqslant 0. \tag{2.6}$$

The thermodynamic inequality in (2.6) is satisfied if we set $\dot{e}_{IN} = \alpha_1 T$ ($\alpha_1 > 0$), which corresponds to the differential viscosity law, or if we set $\dot{e}_{IN} = \alpha_2 S$ ($\alpha_2 > 0$; $S$ is the deviator of the tensor $T$), which corresponds to the associate plastic law $\dot{\chi}_i = -\alpha_3(\partial\Psi/\partial\chi_i)$ ($\alpha_3 > 0$), and if we assume, in particular, that the heat flux $\boldsymbol{q} = -\lambda\tilde{\nabla}\Theta$ (the thermal conductivity $\lambda > 0$) satisfies the Fourier equation (for the assumption for the general case see, for example, in [10]).

In view of relations (2.5), the functional (2.1), (2.3) can be written as

$$W_1 = \frac{1}{2}\int_0^t P_{\mathrm{II}} \cdot\cdot\, \dot{C}_{\varkappa_2}\, d\tau = \int_0^t JT \cdot\cdot\, D_E\, d\tau. \tag{2.7}$$

Relations (2.2) and (2.5) imply that if deformation is due to only inelastic and temperature effects [$\dot{C}_{\varkappa_2} = 0$ ($D_E = 0$) throughout the process], the stress $T$ and the derivative $\partial W_1/\partial\Theta$ vanish. Then, relation (2.6) for the entropy implies that

$$s = -\frac{\partial \Psi_2}{\partial \Theta}, \qquad s\Big|_{\Theta=\Theta_0} = -\frac{\partial \Psi_2}{\partial \Theta}\Big|_{\Theta=\Theta_0} = 0.$$

From the first law of thermodynamics,

$$\rho(\dot{\Psi} + s\dot{\Theta} + \Theta\dot{s}) = T \cdot\cdot\, D + \rho\Omega - \tilde{\nabla} \cdot \boldsymbol{q}, \tag{2.8}$$

where $\Omega$ is the specific rate of heat production by internal sources; for this case (similarly to [11]), we obtain

$$-\rho\Theta\,\frac{\partial^2 \Psi_2}{\partial \Theta^2}\,\dot{\Theta} = \rho\dot{Q},$$

and, hence,

$$-\Theta\,\frac{\partial^2 \Psi_2}{\partial \Theta^2} = \frac{dQ}{d\Theta} = c_T$$

($\rho\dot{Q} = \rho\Omega - \tilde{\nabla} \cdot \boldsymbol{q}$ is the rate of change of the heat transferred to unit mass and $c_T$ is the thermal conductivity of unit mass under zero stress). Assuming that $c_T$ depends only on the temperature and writing this relation as

$$c_T = c_{T_0} + \int_{\Theta_0}^{\Theta} c_{T_1}(\Theta_1)\, d\Theta_1,$$

in view of the above initial conditions, we have

$$\begin{gathered} \frac{\partial \Psi_2}{\partial \Theta} = c_{T_0} \ln \frac{\Theta_0}{\Theta} - \int_{\Theta_0}^{\Theta} \ln\Big(\frac{\Theta}{\Theta_1}\Big) c_{T_1}(\Theta_1)\, d\Theta_1, \\ \Psi_2 = c_{T_0}\Big(\Theta \ln \frac{\Theta_0}{\Theta} + (\Theta - \Theta_0)\Big) - \int_{\Theta_0}^{\Theta} \Big(\Theta \ln \frac{\Theta}{\Theta_1} - (\Theta - \Theta_1)\Big) c_{T_1}(\Theta_1)\, d\Theta_1. \end{gathered} \tag{2.9}$$

As a result, relation (2.6) for entropy becomes

$$s = \frac{J\beta}{\rho_0}\, I_1(T) - \frac{1}{\rho_0}\,\frac{\partial W_1}{\partial \Theta} - c_{T_0} \ln \frac{\Theta_0}{\Theta} + \int_{\Theta_0}^{\Theta} \ln\Big(\frac{\Theta}{\Theta_1}\Big) c_{T_1}(\Theta_1)\, d\Theta_1. \tag{2.10}$$

Reverting to the first law of thermodynamics (2.8) and using the expressions for $\Psi$ and relation (2.9) and (2.10), we determine the internal (intrinsic) dissipation $\varphi = T \cdot\cdot\, D - \rho(\dot{\Psi} + s\dot{\Theta})$. As a result, we have $\varphi =$

$T \cdot\cdot D_{IN} - J^{-1} W_{1,\chi_i} \dot{\chi}_i$. With the use of the Fourier equation for the heat flux, the entropy production can be represented as

$$\rho \Theta \dot{s} = T \cdot\cdot D_{IN} - J^{-1} W_{1,\chi_i} \dot{\chi}_i + \rho \Omega + \tilde{\nabla} \cdot (\lambda \tilde{\nabla} \Theta). \tag{2.11}$$

Entropy is produced by external heat sources [the last two terms on the right of Eq. (2.11)] and latent sources [the first two terms on the right of Eq. (2.11)], which depend on the inelastic deformation power and changes in the internal structure of the material. The fraction of the energy of the latent sources converted to heat is determined from the heat-conduction equation.

By virtue of the principle of equipresence, the arguments for the entropy are the tensor $C_{\varkappa_2}$, the scalar functions $\chi_i$, and the temperature $\Theta$. In this case,

$$\dot{s} = \frac{\partial s}{\partial C_{\varkappa_2}} \cdot\cdot \dot{C}_{\varkappa_2} + \frac{\partial s}{\partial \chi_i} \dot{\chi}_i + \frac{\partial s}{\partial \Theta} \dot{\Theta}.$$

Substituting this expression into the left side of relation (2.11) and taking into account (1.7), we obtain the heat-conduction equation

$$c \dot{\Theta} = \dot{Q}_E + \dot{Q}_{IN} + \rho \Omega + \tilde{\nabla} \cdot (\lambda \tilde{\nabla} \Theta),$$

where $c = \rho \Theta (\partial s / \partial \Theta)$ is the heat capacity and $\dot{Q}_E$ is the rate of heat production by elastic deformations:

$$\dot{Q}_E = -2 \rho \Theta \Big( F \cdot \frac{\partial s}{\partial C_{\varkappa_2}} \cdot F^{\mathrm{t}} \Big) \cdot\cdot D_E;$$

$\dot{Q}_{IN}$ is the rate of heat production by inelastic deformations and structural changes in the material:

$$\dot{Q}_{IN} = T \cdot\cdot D_{IN} - \dot{\chi}_i \Big( J^{-1} W_{1,\chi_i} + \rho \Theta \frac{\partial s}{\partial \chi_i} \Big).$$

The heat-conduction equation obtained by differentiation of relation (2.10) with respect to time is more convenient for work. After simple transformations, we have

$$\begin{aligned} &(\beta_{,\Theta} \Theta I_1(T) + J^{-1} \rho_0 c_T) \dot{\Theta} + \Big[ \beta (I_1(T) I_1(D) + I_1(\dot{T})) - J^{-1} \frac{d}{dt} (W_{1,\Theta}) \Big] \Theta \\ &\quad = T \cdot\cdot D_{IN} - J^{-1} W_{1,\chi_i} \dot{\chi}_i + \rho \Omega + \tilde{\nabla} \cdot (\lambda \tilde{\nabla} \Theta). \end{aligned} \tag{2.12}$$

Here $I_1$ is the first invariant of the correspond tensor and $f_{,\alpha} = \partial f / \partial \alpha$. Taking into account that

$$I_1(\dot{T}) = 2 T \cdot\cdot D - I_1(T) I_1(D) + g \cdot\cdot \Big( 2 J^{-1} F \cdot \frac{d}{dt} \Big( \frac{\partial W_1}{\partial C_{\varkappa_2}} \Big) \cdot F^{\mathrm{t}} \Big),$$

$$\frac{d}{dt} \Big( \frac{\partial W_1}{\partial C_{\varkappa_2}} \Big) = \frac{1}{2} \frac{d P_{\mathrm{II}}}{dt} = \frac{\partial^2 W}{\partial C_E^2} \cdot\cdot \dot{C}_{\varkappa_2} + \frac{1}{2} (P_{\mathrm{II},\chi_i} \dot{\chi}_i + P_{\mathrm{II},\Theta} \dot{\Theta}),$$

$$\frac{d}{dt} \Big( \frac{\partial W_1}{\partial \Theta} \Big) = J T_{,\Theta} \cdot\cdot D_E + \dot{\chi}_i \int_0^t J T_{,\Theta \chi_i} \cdot\cdot D_E \, d\tau + \dot{\Theta} \int_0^t J T_{,\Theta\Theta} \cdot\cdot D_E \, d\tau,$$

from Eq. (2.12), we obtain the following relations for the heat capacity and the rate of heat production by elastic and inelastic deformations and structural changes in the material:

$$c = J^{-1} \rho_0 c_T + \Theta \Big[ (\beta_{,\Theta} + 2\beta^2) I_1(T) + \beta I_1(T_{,\Theta}) - J^{-1} \int_0^t J T_{,\Theta\Theta} \cdot\cdot D_E \, d\tau \Big],$$

$$\dot{Q}_E = \Theta [T_{,\Theta} - 2\beta T - \beta (g \cdot\cdot \tilde{L}_6^{\mathrm{IV}})] \cdot\cdot D_E,$$

$$\dot{Q}_{IN} = (1 - 2\beta\Theta) T \cdot\cdot D_{IN} + \dot{\chi}_i \Big[ J^{-1} \int_0^t J (\Theta(t) T_{,\Theta \chi_i} - T_{,\chi_i}) \cdot\cdot D_E \, d\tau - \beta \Theta I_1(T_{,\chi_i}) \Big].$$

The fourth-rank tensor $\tilde{L}_6^{\rm IV}$ present in these relations defines the properties of the material at the current time. The general expression for this tensor was obtained in [3]:

$$\tilde{L}_6^{\rm IV} = 4J^{-1}F \cdot \Big(F \overset{3}{\circ} \frac{\partial^2 W}{\partial C_E^2} \overset{2}{*} F^{\rm t}\Big) \cdot F^{\rm t}.$$

Here the sign $\overset{\text{“}2\text{”}}{*}$ denotes the scalar multiplication from the right of the second rank tensor (in this case, $F^{\rm t}$) into the second basis vector of the fourth-rank tensor (in this case, $\partial^2 W/\partial C_E^2$).

**3. Limitations Following from the Principle of Objectivity.** Equations (2.4), (2.6), (2.7), and (2.12) contain the powers $T \cdot\cdot\, \dot{e}_E$, $T \cdot\cdot\, \dot{e}_{IN}$, and $T \cdot\cdot\, \dot{e}_\Theta$. Let us consider their transformation under rigid-body rotation of the current, inelastic, temperature, and initial configurations.

In relation (1.8), the gradients $F_\Theta$ transform the initial configuration $\varkappa_0$ into the configuration $\varkappa_1$, which, in turn, is the initial configuration for the gradient $F_{IN}$, which transforms this configuration into the configuration $\varkappa_2$, which is converted to the current configuration $\varkappa$ by the gradient $F_E$:

$$F\colon \varkappa_0 \to \varkappa; \qquad F_\Theta\colon \varkappa_0 \to \varkappa_1; \qquad F_{IN}\colon \varkappa_1 \to \varkappa_2; \qquad F_E\colon \varkappa_2 \to \varkappa.$$

Following [12, 13], we examine how these gradients are transformed under changes in the reference systems with respect to which the motions resulting in the configurations $\varkappa_1$, $\varkappa_2$, and $\varkappa$ are determined, and under a change in the reference system in which the quantities determining the initial configuration $\varkappa_0$ are described. In other words, we examine how the gradients $F$, $F_E$, $F_{IN}$, and $F_\Theta$ change under translation and rigid-body rotation of the configurations $\varkappa$, $\varkappa_2$, $\varkappa_1$, and $\varkappa_0$.

For a change of only the current configuration corresponding to the time $t$ and the remaining configurations unchanged, we have

$$F' = O \cdot F, \qquad F'_E = O \cdot F_E, \qquad F'_{IN} = F_{IN}, \qquad F'_\Theta = F_\Theta. \tag{3.1}$$

Here all gradients are determined at the time $t$. For a rigid-body transformation of only the configuration $\varkappa_2$ corresponding to the time $t$ with the remaining configurations unchanged, we have

$$F' = F, \qquad F'_E = F_E \cdot O^{\rm t}_{IN}, \qquad F'_{IN} = O_{IN} \cdot F_{IN}, \qquad F'_\Theta = F_\Theta. \tag{3.2}$$

For a rigid-body transformation of only the configuration $\varkappa_1$ corresponding to the time $t$ with the remaining configurations unchanged, we obtain the relations

$$F' = F, \qquad F'_E = F_E, \qquad F'_{IN} = F_{IN} \cdot O^{\rm t}_\Theta, \qquad F'_\Theta = O_\Theta \cdot F_\Theta. \tag{3.3}$$

Finally, for a change in the initial configuration $\varkappa_0$, we have

$$F' = F \cdot O_0, \qquad F'_E = F_E, \qquad F'_{IN} = F_{IN}, \qquad F'_\Theta = F_\Theta \cdot O_0. \tag{3.4}$$

In the case of changes in all configurations, relations (3.1)–(3.4) imply

$$F' = O \cdot F \cdot O_0, \qquad F'_E = O \cdot F_E \cdot O^{\rm t}_{IN}, \tag{3.5}$$

$$F'_{IN} = O_{IN} \cdot F_{IN} \cdot O^{\rm t}_\Theta, \qquad F'_\Theta = O_\Theta \cdot F_\Theta \cdot O_0.$$

In view of (3.5), following [13], we elucidate how the powers $T \cdot\cdot\, \dot{e}_E$, $T \cdot\cdot\, \dot{e}_{IN}$, and $T \cdot\cdot\, \dot{e}_\Theta$ change under rigid-body rotation of the configurations $\varkappa$, $\varkappa_2$, $\varkappa_1$, and $\varkappa_0$ under the action of the introduced orthogonal tensors. The true-stress tensor is an objective tensor, and, hence, $T' = O \cdot T \cdot O^{\rm t}$. It has been shown [3] that the elastic and inelastic rate gradients $l_E = (\tilde{\nabla} \boldsymbol{v}_E)^{\rm t}$ and $l_{IN} = (\tilde{\nabla} \boldsymbol{v}_{IN})^{\rm t}$ are written as

$$l_E = \dot{F}_E \cdot F_E^{-1} = \dot{e}_E + \dot{d}_E, \qquad l_{IN} = F_E \cdot \dot{F}_{IN} \cdot F_{IN}^{-1} \cdot F_E^{-1} = \dot{e}_{IN} + \dot{d}_{IN}.$$

Similarly, it can be shown that

$$l_\Theta = F_{IN} \cdot F_E \cdot \dot{F}_\Theta \cdot F_\Theta^{-1} \cdot F_E^{-1} \cdot F_{IN}^{-1} = \dot{e}_\Theta + \dot{d}_\Theta \qquad [l_\Theta = (\tilde{\nabla} \boldsymbol{v}_\Theta)^{\rm t}].$$

In view of relations (3.5), we determine $l'_E$, $l'_{IN}$, and $l'_\Theta$. As a result, we have

$$T' \cdot\cdot\, \dot{e}'_E = T' \cdot\cdot\, l'_E = T \cdot\cdot\, \dot{e}_E + (F_E^{-1} \cdot T \cdot F_E) \cdot\cdot (\dot{O}^{\rm t}_{IN} \cdot O_{IN}),$$

$$T' \cdot\cdot\, \dot{e}'_{IN} = T' \cdot\cdot\, l'_{IN} = T \cdot\cdot\, \dot{e}_{IN} - (F_E^{-1} \cdot T \cdot F_E) \cdot\cdot (\dot{O}^{\rm t}_{IN} \cdot O_{IN}) + (F_{IN}^{-1} \cdot F_E^{-1} \cdot T \cdot F_E \cdot F_{IN}) \cdot\cdot (\dot{O}^{\rm t}_\Theta \cdot O_\Theta), \tag{3.6}$$

$$T' \cdot\cdot\, \dot{e}'_\Theta = T' \cdot\cdot\, l'_\Theta = T \cdot\cdot\, \dot{e}_\Theta - (F_{IN}^{-1} \cdot F_E^{-1} \cdot T \cdot F_E \cdot F_{IN}) \cdot\cdot (\dot{O}_\Theta^{\mathrm{t}} \cdot O_\Theta)$$

$$+ (F_\Theta^{-1} \cdot F_{IN}^{-1} \cdot F_E^{-1} \cdot T \cdot F_E \cdot F_{IN} \cdot F_\Theta) \cdot\cdot (\dot{O}_0 \cdot O_0^{\mathrm{t}}).$$

The last term in the third expression vanishes since the tensor $O_0$ defining the initial anisotropy of the material does not depend on time.

From the above relations it follows that the total power is an invariant quantity: $T' \cdot\cdot\, \dot{e}' = T \cdot\cdot\, \dot{e}$ [the Noll axiom (see [9])]. However, the powers of elastic deformation and mechanical and thermal dissipation depend on the rigid transformations of the configurations $\varkappa_2$ and $\varkappa_1$ due to the terms containing double scalar multiplication by skew-symmetric tensors (spins) $A_{IN} = \dot{O}_{IN}^{\mathrm{t}} \cdot O_{IN}$ and $A_\Theta = \dot{O}_\Theta^{\mathrm{t}} \cdot O_\Theta$. If these terms do not vanish, by an appropriate choice of the tensors $O_{IN}$ and $O_\Theta$, one can obtain powers of elastic deformation and mechanical and thermal dissipation of arbitrary magnitude and sign due only to rigid changes in the reference configurations. These terms vanish in two cases: 1) if the rotation tensors of the spins $A_{IN}$ and $A_\Theta$ are symmetric; 2) if $R_{IN} = g$ and $R_\Theta = g$ at any time [$R_{IN}$ and $R_\Theta$ are the orthogonal tensors in the polar decompositions of the site gradients $F_{IN}$ and $F_\Theta$, respectively]. The first condition is satisfied only for a purely elastic process with initial isotropy of the material (see [13]). In the case of an elastic–inelastic process, this condition is not satisfied, as follows from expressions (2.5). Since relations (3.6) are valid for any orthogonal tensors $O_{IN}$ and $O_\Theta$, then, setting $O_{IN} = R_{IN}$ and $O_\Theta = R_\Theta$, we obtain the second condition. As a result, the total site gradient is represented as $F = F_E \cdot U_{IN} \cdot U_\Theta$, where $U_{IN}$ and $U_\Theta$ are symmetric positive definite tensors of pure strains in the polar decomposition of the site gradients $F_{IN} = R_{IN} \cdot U_{IN}$, $F_\Theta = R_\Theta \cdot U_\Theta$, and $R_{IN} = g$, $R_\Theta = g$. Thus, we proved the necessary invariance condition for the power of elastic deformation and mechanical and thermal dissipation under rigid transformations of the reference configurations, which is also a sufficient condition. Indeed, provided that $R_{IN} = g$ and $R_\Theta = g$ and that only the current configuration changes, we have

$$F' = O \cdot F, \quad F'_E = O \cdot F_E, \quad U'_{IN} = U_{IN}, \qquad U'_\Theta = U_\Theta.$$

For rigid-body transformation of only the configuration $\varkappa_2$, we obtain

$$F' = F, \qquad F'_E = F_E, \qquad U'_{IN} = U_{IN}, \qquad U'_\Theta = U_\Theta,$$

and for rigid-body transformation of only the configuration $\varkappa_1$, we obtain

$$F' = F, \qquad F'_E = F_E, \qquad U'_{IN} = U_{IN}, \qquad U'_\Theta = U_\Theta.$$

Finally, for a change in the initial configuration $\varkappa_0$, we have

$$F' = F \cdot O_0, \qquad F'_E = F_E \cdot O_0, \qquad U'_{IN} = O_0^{\mathrm{t}} \cdot U_{IN} \cdot O_0, \qquad U'_\Theta = O_0^{\mathrm{t}} \cdot U_\Theta \cdot O_0.$$

As a result, for changes in all configurations, we obtain

$$F' = O \cdot F \cdot O_0, \qquad F'_E = O \cdot F_E \cdot O_0,$$

$$U'_{IN} = O_0^{\mathrm{t}} \cdot U_{IN} \cdot O_0, \qquad U'_\Theta = O_0^{\mathrm{t}} \cdot U_\Theta \cdot O_0.$$

Using these expressions and calculating (with allowance for the time independence of $O_0$) the quantities

$$l'_E = \dot{F}'_E \cdot (F'_E)^{-1}, \qquad l'_{IN} = F'_E \cdot \dot{U}'_{IN} \cdot (U'_{IN})^{-1} \cdot (F'_E)^{-1},$$

$$l'_\Theta = U'_{IN} \cdot F'_E \cdot \dot{U}'_\Theta \cdot (U'_\Theta)^{-1} \cdot (F'_E)^{-1} \cdot (U'_{IN})^{-1},$$

we find that the powers of elastic deformation and mechanical and thermal dissipation are invariant under rigid transformations of the reference configurations. This statement implies objectivity of all relations in Sec. 2.

**Conclusions.** Within the framework of the kinematics determined by the imposition of elastic–inelastic site gradients (which transform an intermediate configuration into a close current configuration) onto finite elastic–inelastic site gradients (which transform the initial configuration into an intermediate configuration), similarly to [3], we derived a representation of the total site gradient in terms of an elastic, inelastic, and temperature site gradients that coincides in shape with the well-known Lie representation but is free from the drawbacks of the latter. It was shown that by virtue of the principle of objectivity, as in the Lie representation, the inelastic and temperature site gradients should be pure deformations without rotations.

Stress and entropy relations based on thermodynamics were derived and a heat-conduction equation was constructed using the functional introduced in [3] as one of the terms in the free energy expression. In this functional, the constants appearing in the fourth-rank tensor that defines the material properties at the current time and depends only on the elastic kinematics, were assumed to be functions of temperature and the scalar structural parameters determined by the inelastic kinematics.

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